

# WEIGHTED POLYNOMIALS AND WEIGHTED PLURIPOTENTIAL THEORY

BY

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ABSTRACT. Let  $E$  be a compact subset of  $\mathbb{C}^N$  and  $w \geq 0$  a weight function on  $E$  with  $w > 0$  on a non-pluripolar subset of  $E$ . To  $(E, w)$  we associate a canonical circular set  $Z \subset \mathbb{C}^{N+1}$ . We obtain precise relations between the weighted pluricomplex Green function and equilibrium measure of  $(E, w)$  and the pluricomplex Green function and equilibrium measure of  $Z$ . These results, combined with an appropriate form of the Bernstein-Markov inequality, are used to obtain asymptotic formulas for the leading coefficients of orthonormal polynomials with respect to certain exponentially decreasing weights in  $\mathbb{R}^N$ .

## INTRODUCTION

An admissible weight on a compact set  $E \subset \mathbb{C}^N$  is a function  $w \geq 0$  which is strictly positive on a non-pluripolar subset of  $E$ . Associated to  $(E, w)$  is a weighted pluripotential theory involving weighted polynomials, i.e, functions of the form  $w^d p$  where  $p$  is a polynomial of degrees  $\leq d$ , a weighted pluricomplex Green function  $V_{E,Q}$  and a weighted equilibrium measure  $d\mu_{eq}(E, w)$ . The definitions of these concepts are given in section 1.

In the one-dimesional case ( $N = 1$ ) the book of Saff and Totik [SaTo] has many basic results. In the one-dimensional case, weighted polynomials arise in diverse problems – approximation theory, orthognal polynomials, random matrices, statistical physics. For an example of recent developments see [Dei].

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In the higher dimensional case, weighted pluripotential theory was used in [BL2] to obtain results on directional Tchebyshev constants of compact sets – the main procedure being an inductive step from circular compact sets to a weighted problem in one less variable.

In this paper we further develop the relation between weighted pluripotential theory on a compact set  $E \subset \mathbb{C}^N$  with admissible (see (1.10)) weight  $w$  and the potential theory of a canonically associated circular set  $Z \subset \mathbb{C}^{N+1}$  (defined in (2.1)).

We show that  $V_Z$ , the pluricomplex Green function of  $Z$ , and  $d\mu_{eq}(Z)$  the equilibrium measure of  $Z$ , are related to the weighted pluricomplex Green function and the weighted equilibrium measure of  $E$  with weight  $w$ .

The main results are:

**Theorem 2.1.**

$$V_Z = (V_{E,Q}) \circ L + \log |t| \quad \text{for } t \neq 0$$

**Theorem 2.2.**  $L_* \left( \frac{1}{2\pi} d\mu_{eq}(Z) \right) = d\mu_{eq}(E, w).$

Here  $t$  is the first coordinate of  $\mathbb{C}^{N+1}$  and  $L : \{\mathbb{C}^{N+1} - \{(t, z) | t = 0\}\} \rightarrow \mathbb{C}^N$  is given by (2.4).  $L_*$  is the push-forward of measures under  $L$ .

Special cases of the above results may be found in the paper of DeMarco [DeM]. In particular theorem 2.1 generalizes examples of section 4 of [DeM] and theorem 2.2 generalizes lemma 2.3 of [DeM].

The advantage of considering weighted pluripotential theory is that (up to a limiting procedure described in section 5 of this paper) the potential theory of a *general* compact circular set in  $(N + 1)$  variables may be reduced to the weighted case in  $N$  variables.

In section 3 we consider the Bernstein-Markov (B-M) inequality (for the definition see (3.1)). This inequality may be used to relate asymptotics of orthonormal

polynomials with respect to a measure  $\mu$  on  $E$  to potential theoretic invariants of  $E$ . We introduce a weighted version of the B-M inequality (see(3.2)). We show (theorem 3.1) that the weighted B-M inequality holds on  $E$  with weight  $w$  and measure  $\mu$  if and only if the B-M inequality holds for an associated measure on  $Z$ . Then we give the following cases where the B-M inequality holds.

**Corollary 3.1.**  $d\mu_{eq}(E, w)$  for  $E$  regular and weight  $w$ .

**Theorem 3.2.**  $(E, w, \sigma)$  where  $E \subset \mathbb{R}^N$  and  $(E, \sigma)$  satisfies the B-M inequality.

In section 4 we obtain asymptotics for the leading coefficients of orthonormal polynomials with respect to certain exponentially decreasing measures on  $\mathbb{R}^n$ . As in the known procedure in the one-variable case, we first scale the problem to obtain a problem on the asymptotics of weighted polynomials. Using the weighted B-M inequality (a special case of theorem 3.2) gives the asymptotics (see example 4.1 and equation (4.24)).

## 1. PRELIMINARIES

Let  $E$  be a bounded subset of  $\mathbb{C}^N$ . The pluricomplex Green function of  $E$  is defined by

$$(1.1) \quad V_E(z) := \sup\{u(z) | u \in \mathcal{L}, u \leq 0 \text{ on } E\} \text{ where}$$

$$(1.2) \quad \mathcal{L} = \{u | u \text{ is plurisubharmonic (PSH) on } \mathbb{C}^N, u(z) \leq \log^+ |z| + C\}.$$

is the Lelong class of PSH functions of logarithmic growth. (We use the notation

$$|z| := \left( \sum_{i=1}^N |z_i|^2 \right)^{1/2} \text{ for } z = (z_1, \dots, z_N) \in \mathbb{C}^N.$$

A set  $E \subset \mathbb{C}^N$  is said to be *pluripolar* if for all points  $a \in E$ , there is a neighborhood  $U$  of  $a$  and a function  $v$  which is PSH on  $U$  such that  $E \cap U \subset \{z \in U | v(z) =$

$-\infty\}$ . A property of a set  $E$  is said to hold *quasi-everywhere* (q.e.) if there is a pluripolar set  $P \subset E$  and the property holds at all points of  $E \setminus P$ .

For  $G$  an open subset of  $\mathbb{C}^N$  and  $f$  a real-valued function on  $G$ , we let  $f^*$  denote its uppersemicontinuous (u.s.c.) regularization, defined by

$$f^*(z) = \overline{\lim}_{\xi \rightarrow z} f(\xi) \quad \text{for } z \in G.$$

$V_E^* \in \mathcal{L}$  if and only if  $E$  is non-pluripolar [K]. For  $E$  non-pluripolar, the equilibrium measure of  $E$  is defined by

$$(1.3) \quad d\mu_{eq} = d\mu_{eq}(E) := (dd^c V_E^*)^N$$

where  $(dd^c)^N$  is the complex Monge-Ampère operator.  $d\mu_{eq}$  is a positive Borel measure of total mass  $(2\pi)^N$  and with  $\text{supp}(d\mu_{eq}) \subset \overline{E}$ . [K].

In the case that  $E$  is a compact set, it is a result of Siciak and Zaharyuta ([K], theorem 5.1.7) that

$$(1.4) \quad V_E(z) = \log \phi_E(z)$$

where

$$(1.5) \quad \phi_E(z) = \sup\{|p(z)|^{\frac{1}{\deg(p)}} \mid p \text{ is a holomorphic polynomial, } \deg(p) \geq 1 \text{ and } \|p\|_E \leq 1\}.$$

It follows that  $V_E(z) = V_{\hat{E}}(z)$  where  $\hat{E}$  denotes the polynomially convex hull of  $E$ .

We will also use the class  $\mathcal{H}$  of logarithmically homogeneous PSH functions on  $\mathbb{C}^N$  defined by

$$(1.6) \quad \mathcal{H} := \{u \in \mathcal{L} \mid u(tz) = u(z) + \log |t| \text{ for all } z \in \mathbb{C}^N, t \in \mathbb{C}\}.$$

For  $E$  a bounded set in  $\mathbb{C}^n$ , we define

$$(1.7) \quad H_E(z) := \sup\{u(z) \mid u \in \mathcal{H}, u \leq 0 \text{ on } E\}.$$

For  $E$  compact (see [Si2]) we have

$$(1.8) \quad H_E(z) = \log \psi_E(z) \text{ where}$$

$$(1.9)$$

$$\psi_E(z) = \sup\{|p(z)|^{\frac{1}{\deg(p)}} \mid p \text{ is a homogeneous holomorphic polynomial, } \deg p \geq 1 \text{ and } \|p\|_E \leq 1\}.$$

For  $E$  a compact set in  $\mathbb{C}^N$ , an *admissible weight function* is a real-valued function on  $E$  satisfying

$$(1.10) \quad \begin{aligned} &i) \quad w \geq 0 \\ &ii) \quad w \text{ is u.s.c.} \\ &iii) \quad \{z \in E \mid w(z) > 0\} \text{ is non-pluripolar.} \end{aligned}$$

In particular, if  $E$  admits on admissible weight function then  $E$  itself is non-pluripolar.

There is a “weighted” version of the pluricomplex Green function (see [S.1], [SaTo, appendix B]) defined as follows: let

$$(1.11) \quad Q := -\log w$$

Then  $Q$  is lowersemicontinuous (l.s.c.) on  $E$ . The weighted pluricomplex Green function of  $E$  with weight  $w$  is defined by

$$(1.12) \quad V_{E,Q} := \sup\{u(z) \mid u \in \mathcal{L}, u \leq Q \text{ on } E\}$$

The weighted equilibrium measure of  $E$  is defined by

$$(1.13) \quad d\mu_{eq}(E, w) := (dd^c V_{E,Q}^*)^N.$$

It is a positive Borel measure with  $\text{supp}(d\mu_{eq}(E, w)) \subset E$ . and total mass  $(2\pi)^N$ .

A weighted polynomial on  $E$  is defined to be a function of the form  $w^d p$  where  $d$  is an integer  $\geq 0$  and  $p$  is a holomorphic polynomial of degree  $\leq d$ . Note that

if  $\|w^d p\|_E \leq 1$  then  $\frac{1}{d} \log |p(z)| \leq Q(z)$  on  $E$  and since  $\frac{1}{d} \log |p(z)| \in \mathcal{L}$  we have,  $\frac{1}{d} \log |p(z)| \leq V_{E,Q}(z)$  for all  $z \in \mathbb{C}^N$ .

It is known (see [Si1] or [SaTo], appendix B) that

$$(1.14) \quad V_{E,Q}(z) = \log \phi_{E,Q}(z) \quad \text{where}$$

$$(1.15)$$

$$\phi_{E,Q}(z) = \sup\{|p(z)|^{\frac{1}{d}} \mid \|w^d p\|_E \leq 1, \deg p \geq 1 \text{ and } w^d p \text{ is a weighted polynomial}\}.$$

A set  $E$  is defined to be *regular* if  $V_E$  is continuous on  $\mathbb{C}^N$ . A set  $E$  is defined ([Si1]) to be *locally regular at a point*  $a \in \overline{E}$  if for each  $r > 0$ ,  $V_{E \cap B(a,r)}$  is continuous at  $a$ . Here  $B(a, r) := \{z \in \mathbb{C}^N \mid |z - a| \leq r\}$  denotes the ball center  $a$ , radius  $r$ . It is sufficient, for  $E$  to be locally regular at  $a$ , that  $V_{E \cap B(a,r)}$  be continuous at  $a$  for all  $r > 0$  sufficiently small.

$E$  is said to be *locally regular* if it is locally regular at each point of  $\overline{E}$ .

For  $E$  compact and locally regular and  $w$  a continuous admissible weight function on  $E$  then  $V_{E,Q}$  is continuous [Si1].

For  $u \in \mathcal{L}$  we define its Robin function  $\rho_u$  by

$$(1.16) \quad \rho_u(z) := \overline{\lim_{\substack{|s| \rightarrow +\infty \\ s \in \mathbb{C}}} u(sz) - \log |s|$$

Then  $\rho(z) \in \mathcal{H}$ .

## 2. EQUILIBRIUM MEASURES

Let  $E$  be a compact set in  $\mathbb{C}^N$  and  $w$  an admissible weight function on  $E$ . We associate the set  $Z = Z(E, w) \subset \mathbb{C}^{N+1}$  defined as follows:

$$(2.1)$$

$$Z := \{(t, \lambda_1 t, \dots, \lambda_N t) \in \mathbb{C}^{N+1} \mid (\lambda = \lambda_1, \dots, \lambda_N) \in E, t \in \mathbb{C} \text{ and } |t| = w(\lambda)\}$$

We will relate the weighted potential theory on  $E$  with weight  $w$  to potential theory on  $Z$ . We will use the notation  $(t, z)$  for a point in  $\mathbb{C}^{N+1}$  where  $t \in \mathbb{C}$  and  $z \in \mathbb{C}^N$ . We denote by  $C_\lambda$  the complex line in  $\mathbb{C}^{N+1}$  given by

(2.2)

$$C_\lambda := \{(t, z) \in \mathbb{C}^{N+1} \mid z_j = \lambda_j t \text{ for } j = 1, \dots, N \text{ with } (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N \text{ and } t \in \mathbb{C}\}$$

$Z$  is a circular set, i.e., if  $(t, z) \in Z$  then  $(e^{i\theta}t, e^{i\theta}z) \in Z$  for all  $\theta \in [0, 2\pi)$ .  $\bar{Z}$  (the closure of  $Z$ ) is compact and circular.

$Z$  is non-pluripolar since  $E$  is non-pluripolar ([BL2], lemma 6.1). Note that for  $w \equiv 1$  (the “unweighted” case) then  $Z = \{|t| = 1\} \times E$

Since  $Z \subset \bigcup_{\lambda \in E} C_\lambda$  the same is true for  $\bar{Z}$  and so  $\bar{Z} \cap \{(t, z) \in \mathbb{C}^{N+1} \mid t = 0\}$  is either empty (if  $w$  is bounded below, on  $E$ , by a positive constant) or else consists only of the origin.

**Proposition 2.1.**  $H_Z = H_{\bar{Z}}$  and  $V_Z = V_{\bar{Z}}$

**Proof:** To prove the first statement it suffices to show that if  $u \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$ ,  $u \leq 0$  on  $Z$  then  $u \leq 0$  on  $\bar{Z}$ . But since  $w$  is u.s.c. we have, for  $\lambda \in E$ :

$$(2.3) \quad \bar{Z} \cap C_\lambda \subset \{(t, z) \in \mathbb{C}_\lambda \mid |t| \leq w(\lambda)\}$$

Applying the maximum principle to the subharmonic function  $t \rightarrow u(t, \lambda_1 t, \dots, \lambda_N t)$  we have  $u \leq 0$  on  $\bar{Z}$ .

The second statement follows similarly.

**Proposition 2.2.**  $V_Z = \text{Max}(0, H_Z)$  and  $\rho_{V_Z} = H_Z^*$ .

The first statement follows from ([Si1], proposition 5.6) and the second from homogeneity (see [BLM], lemma 5.1).

**Proposition 2.3.**  $d\mu_{eq}(Z)$  has compact support in  $\mathbb{C}^{N+1} - \{t = 0\}$ .

**Proof:** Since  $H_Z^* \in \mathcal{H}$  and  $H_Z^*(0) = -\infty$  then  $H_Z^* < 0$  in a neighborhood of the origin (estimates on the size of that neighborhood, known as the Sibony-Wong inequality, can be found in ([A], [Si2])).

Thus, the origin is an interior point of  $\widehat{Z}$ . But  $d\mu_{eq}(Z)$  places no mass on the interior of  $\widehat{Z}$  so the result follows.  $\square$

Let  $L$  denote the mapping  $L : \mathbb{C}^{N+1} - \{t = 0\} \rightarrow \mathbb{C}^N$  given by

$$(2.4) \quad L(t, z) = \frac{z}{t} := \lambda \in \mathbb{C}^N$$

If we consider  $\mathbb{P}^N$  (complex projective  $N$ -space) as the space of lines through the origin in  $\mathbb{C}^{N+1}$ , then  $L$  gives one of the standard coordinate charts for  $\mathbb{P}^N$ . Note that  $L(C_\lambda) = \lambda$ .

We recall the “ $H$ -principle” of Siciak [Si3]. There is a natural 1 – 1 correspondence between  $\mathcal{H}(\mathbb{C}^{N+1})$  and  $\mathcal{L}(\mathbb{C}^N)$  as follows: To  $\tilde{u}(t, z) \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$  associate

$$(2.5) \quad u(z) := \tilde{u}(1, z)$$

Then  $u \in \mathcal{L}(\mathbb{C}^N)$ . Conversely, given  $u \in \mathcal{L}(\mathbb{C}^N)$  we let

$$(2.6) \quad \tilde{u}(t, z) := u\left(\frac{z}{t}\right) + \log |t| = L^*u(\lambda) + \log |t| \text{ for } t \neq 0$$

and

$$(2.7) \quad \tilde{u}(0, z) = \overline{\lim_{\substack{|s| \rightarrow +\infty \\ s \in \mathbb{C}}} u(sz) - \log(s) = \rho_u(z)$$

Then  $\tilde{u} \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$ .

Furthermore, let  $P_d(t, z)$  be a homogeneous polynomial of degree  $d$  on  $\mathbb{C} \times \mathbb{C}^N$ . Then  $\frac{1}{d} \log |P_d(t, z)| \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$  and the associated element of  $\mathcal{L}(\mathbb{C}^N)$  under (2.5) is  $\frac{1}{d} \log |P_d(1, z)|$ .  $P_d(1, z)$  is, of course a polynomial in  $z$  of degree  $\leq d$ .

Conversely, given a polynomial  $G_d(z)$  in  $z$  of degree  $\leq d$ , then  $\frac{1}{d} \log |G_d(z)| \in \mathcal{L}(\mathbb{C}^N)$ . The associated (via (2.6)) function in  $\mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$  is  $\frac{1}{d} \log |t^d G_d(z/t)|$ .

Note that  $t^d G_d(\frac{z}{t})$  is a homogeneous polynomial on  $\mathbb{C}^{N+1}$  of degree  $d$  in  $(t, z)$ . We use the notation:

$$(2.8) \quad P_d(t, z) := t^d G_d(z/t)$$

Given a weighted polynomial  $w^d G_d(\lambda)$  on  $E$  we can relate its norm on  $E$  with the norm of the associated polynomial  $P_d(t, z)$  on  $Z$  (or equivalently,  $\overline{Z}$ ). Specifically, we have

**Lemma 2.1.**

$$\|w^d G_d\|_E = \|P_d(t, z)\|_Z$$

**Proof:** For  $(t, z) \in Z \cap C_\lambda$  and  $\lambda \in E$ , then

$$(2.9) \quad P_d(t, z) = t^d P_d(1, z/t) = t^d G_d(\lambda) \text{ so}$$

$$(2.10) \quad |P_d(t, z)| = |t|^d |P_d(1, z/t)| = w(\lambda)^d |G_d(\lambda)|$$

The result follows.  $\square$

Theorem 2.1 below gives the relation between the pluricomplex Green function on  $E$  and the homogeneous pluricomplex Green function of  $Z$ .

**Theorem 2.1.**  $H_Z(t, z) = L^*(V_{E,Q}) + \log |t|$  for  $t \neq 0$

**Proof:** Let  $\tilde{u}(t, z) \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$  and suppose  $\tilde{u} \leq 0$  on  $Z$ . Now  $\tilde{u}(t, \lambda_1 t, \dots, \lambda_N t) = \tilde{u}(1, \lambda) + \log |t|$  so for  $\lambda \in E$ , we have,  $\log |t| + \tilde{u}(1, \lambda) \leq 0$  on  $C_\lambda \cap Z$ . Thus,

$$(2.11) \quad u(\lambda) \leq -\log |t| = Q(\lambda) \text{ on } C_\lambda \cap Z$$

Hence  $u \leq V_{E,Q}$  and so, using (2.6)

$$\tilde{u}(t, z) \leq L^*(V_{E,Q}) + \log |t| \text{ for } t \neq 0$$

.

Taking the pointwise sup in  $(t, z)$  over all such  $\tilde{u}$  we have

$$(2.12) \quad H_Z \leq L^*(V_{E,Q}) + \log |t| \text{ for } t \neq 0$$

It remains to prove the reverse inequality. Given  $u \in \mathcal{L}(\mathbb{C}^N)$  with  $u \leq Q$  on  $E$  then for  $(t, z) \in C_\lambda \cap Z$ , using (2.6), we have  $\tilde{u}(t, z) = \log |t| + u(\lambda) \leq \log |t| - \log w(\lambda) \leq 0$ . Hence  $\tilde{u}(t, z) \leq 0$  on  $Z$  and  $\tilde{u}(t, z) \leq H_Z$  on  $\mathbb{C} \times \mathbb{C}^N$ . That is,

$$(2.13) \quad L^*(u) + \log |t| \leq H_Z \text{ for } t \neq 0$$

Taking the pointwise sup over all such  $u$  gives the reverse inequality to (2.12).  $\square$

**Corollary 2.1.**  $H_Z^*(t, z) = \widetilde{V_{E,Q}^*}(t, z)$

**Proof:** Consider a point  $(t_0, z_0)$  with  $t_0 \neq 0$ . Let  $\lambda_0 := \frac{z_0}{t_0}$

Then, by theorem 2.1,

$$\overline{\lim}_{(t,z) \rightarrow (t_0, z_0)} H_Z(t, z) = \overline{\lim}_{\lambda \rightarrow \lambda_0} V_{E,Q}(\lambda) + \log |\lambda_0|$$

and the right side is  $\widetilde{V_{E,Q}^*}(t_0, z_0)$  by (2.6). This proves the result for  $t \neq 0$  but since both sides (in the statement of corollary 2.1) are PSH functions on  $\mathbb{C}^{N+1}$  and agree for  $t \neq 0$  they must agree on  $\mathbb{C}^{N+1}$ .  $\square$

Note that the result in ([K], prop. 2.9.16) is similar but not immediately applicable.

Corollaries 2.2, 2.3 and 2.4 deal with the converge of sequences of pluricomplex Green functions for sequences of weights converging in various manners (see also lemma 7.3 [BL2]).

**Corollary 2.2.** *Let  $E \subset \mathbb{C}^N$  be compact and  $\{w_j\}_{j=1,2,\dots}$  a sequence of admissible weights on  $E$  and let  $w$  also be an admissible weight on  $E$ . Suppose that  $w_j \downarrow w$ . Then*

$$\lim_j V_{E,Q_j} = V_{E,Q}.$$

**Proof:** Let  $Z_j := Z_j(E, w_j)$  and  $Z := Z(E, w)$  be the associated circular sets in  $\mathbb{C}^{N+1}$ . But  $\{(t, z) \mid |t| \leq w_j\} \downarrow \{(t, z) \mid |t| \leq w\}$ . Hence  $\hat{Z}_j \downarrow \hat{Z}$  and the result follows from theorem 2.1 and ([K], corollary 5.1.2).

**Corollary 2.3.** *Let  $E, \{w_j\}, w$  be as in corollary 2.2, except that  $w_j \downarrow w$  q.e. then*

$$\lim_j V_{E,Q_j}^* = V_{E,Q}^*$$

**Proof:**  $\bigcap_j \{(t, z) \mid |t| \leq w_j\}$  and  $\{(t, z) \mid |t| \leq w\}$  differ by a pluripolar set so the result follows from ([K], cor. 5.2.5).

**Corollary 2.4.** *Let  $E, \{w_j\}, w$  be as in corollary 2.2 except that  $w_j \uparrow w$  q.e. Then*  

$$\lim_j V_{E,Q_j}^* = V_{E,Q}^*.$$

**Proof:** For some pluripolar set  $F$  we have  $Z_j \cup F \uparrow Z \cup F$ . Hence by ([K], cor. 5.2.5 and 5.2.6)

$$V_{Z_j}^* = V_{Z_j \cup F}^* \downarrow V_{Z \cup F}^* = V_Z^*$$

Hence, using homogeneity,  $H_Z^* \downarrow H_Z$  and by corollary 2.1,

$$V_{E,Q_j}^* \downarrow V_{E,Q}^*. \quad \square$$

By proposition 2.3, we may consider  $L_*(d\mu_{eq}(Z))$ -the push forward of the equilibrium measure  $d\mu_{eq}(Z)$  under  $L$ . Since  $\text{supp}(d\mu_{eq}(Z)) \subset Z \subset \bigcup_{\lambda \in E} \mathbb{C}_\lambda$  we have  $\text{supp}(L_*(d\mu_{eq}(Z))) \subset E$ . There is however a more precise relation. Assume that  $Z$  is regular. The equilibrium measure on  $Z$  and the weighted equilibrium measure on  $E$  are related by:

**Theorem 2.2.**  $L_*\left(\frac{1}{2\pi}d\mu_{eq}(Z)\right) = d\mu_{eq}(E, w).$

**Proof:** The proof is based on lemma 3.3 in [DeM] which itself is based on work of Briend. (Note that we use the convention of Klimek's book [K] for  $d^c := i(\bar{\partial} - \partial)$  not that of [DeM]. This results in the factor  $\frac{1}{2\pi}$  in the statement of theorem 2.2).

$H_Z$  is continuous by proposition 2.2 so, as a consequence of Theorem 2.1  $V_{E,Q}$  is continuous. Then

$$(2.14) \quad dd^c H_Z = dd^c L^*(V_{E,Q}) = L^*(dd^c V_{E,Q}) \text{ for } t \neq 0 \text{ and so}$$

$$(2.15) \quad (dd^c H_Z)^N = L^*(dd^c V_{E,Q})^N = L^*(d\mu_{eq}(E, w)) \text{ for } t \neq 0.$$

Let  $\phi$  be a smooth compactly supported function on  $\mathbb{C}^{N+1} - \{t = 0\}$ . then

$$(2.16) \quad \int_{\mathbb{C}^{N+1}} \phi d\mu_{eq}(Z) = \int_{\mathbb{C}^{N+1}} \phi (dd^c V_Z)^{N+1} = \int_{\mathbb{C}^{N+1}} V_Z dd^c \phi \wedge (dd^c V_Z)^N$$

Now, as  $\epsilon \downarrow 0$ ,  $\text{Max}(H_Z, \epsilon) - \epsilon \uparrow V_Z$  uniformly on  $C^N$ . so

$$(2.17) \quad \int_{\mathbb{C}^{N+1}} V_Z dd^c \phi \wedge (dd^c V_Z)^N = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}^{N+1}} (\text{Max}(H_Z, \epsilon) - \epsilon) dd^c \phi \wedge (dd^c V_Z)^N$$

Note that  $\{z \in \mathbb{C}^{N+1} \mid \text{Max}(H_Z, \epsilon) = \epsilon\}$  is a neighborhood of  $\bar{Z}$  in  $\mathbb{C}^{N+1}$  and if  $\text{Max}(H_Z(z), \epsilon) > \epsilon$  then  $V_Z = H_Z$ . So, in the expression on the right of (2.17) we may replace  $V_Z$  by  $H_Z$  to obtain.

$$(2.18) \quad \int_{\mathbb{C}^{N+1}} V_Z dd^c \phi \wedge (dd^c H_Z)^N = \int_{\mathbb{C}^{N+1}} V_Z dd^c \phi \wedge L^*(dd^c V_{E,Q})^N$$

The integrands in (2.16), (2.17) and (2.18) all have compact support in  $\mathbb{C}^{N+1} - \{t = 0\}$  so the right side of (2.18) is equal to

$$(2.19) \quad \int_{\mathbb{C}^N} \left( \int_{C_\lambda} V_Z dd^c \phi \right) d\mu_{eq}(E, w) = \int_{\mathbb{C}^N} \left( \int_{C_\lambda} \phi dd^c V_Z \right) d\mu_{eq}(E, w)$$

For  $\lambda \in E$ , we let  $dm_\lambda$  be the Lebesgue measure on the circle  $|t| = w(\lambda)$  in  $C_\lambda$  normalized to have total mass 1. Then  $\frac{1}{2\pi} dd^c V_{Z/C_\lambda} = dm_\lambda$ .

The right side of (2.19) is thus equal to

$$(2.20) \quad \frac{1}{2\pi} \int_{\mathbb{C}^N} \left( \int_{C_\lambda} \phi dm_\lambda \right) d\mu_{eq}(E, w)$$

which proves theorem 2.2.  $\square$

The next corollary shows that the assumption of  $\bar{Z}$  being regular may be dropped from the hypothesis of theorem 2.2.

**Corollary 2.5.**  $L_* \left( \frac{1}{2\pi} d\mu_{eq}(Z) \right) = d\mu_{eq}(E, w).$

**Proof:** We need only find a sequence of locally regular compact sets  $E_j$ , admissible, continuous, weights  $w_j$  on  $E_j$  such that  $E_j \downarrow E$  and  $w_j \downarrow w$  on  $E$ . Then  $Z(E_j, w_j) \downarrow Z(E, w)$ . Applying theorem 2.2 to each  $Z(E_j, w_j)$  and  $E_j$  and taking limits gives the result.

To construct such a sequence of  $E_j$  and  $w_j$  we may follow the procedure of ([BL2], section 7).

### 3. THE BERNSTEIN-MARKOV INEQUALITY

Given a compact set  $E \subset \mathbb{C}^N$  and a finite positive Borel measure  $\mu$  on  $E$ , we say that  $(E, \mu)$  satisfies the Bernstein-Markov (B-M) inequality if, for every  $\epsilon > 0$ , there exists a constant  $C = C(\epsilon) > 0$  such that, for all holomorphic polynomials  $p$  we have

$$(3.1) \quad \|p\|_E \leq C(1 + \epsilon)^{\deg(p)} \|p\|_{L^2(\mu)}$$

This inequality may be used to relate  $L^2$  properties of polynomials with potential theoretic invariants of  $E$  (see [B1] and [BL2] for conditions under which the inequality holds).

We will introduce a “weighted” version of the B-M inequality.

Given a compact set  $E \subset \mathbb{C}^N$ , an admissible weight  $w$  on  $E$  and a finite positive Borel measure  $\mu$  on  $E$ , we say that  $(E, w, \mu)$  satisfies the weighted B-M inequality if for all  $\epsilon > 0$ , that exists a constant  $C = C(\epsilon) > 0$  such that, for all weighted polynomials  $w^d p$  we have

$$(3.2) \quad \|w^d p\|_E \leq C(1 + \epsilon)^d \|w^d p\|_{L^2(\mu)}$$

Of course, for  $w \equiv 1$ , (3.2) reduces to (3.1).

We will relate the weighted B-M inequality for  $(E, w, \mu)$  to a B-M inequality on  $\bar{Z}$  with respect to a certain associated measure  $\nu$ . The measure  $\nu$  is defined as follows:

$$(3.3) \quad d\nu = dm_\lambda \otimes d\mu \text{ for } \lambda \in E \text{ so that } \text{supp}(\nu) \subset \bigcup_{\lambda \in E} C_\lambda$$

That is, for  $\phi$  continuous with compact support in  $\mathbb{C}^{N+1} - \{t = 0\}$  we have

$$(3.4) \quad \int_{\mathbb{C}^{N+1}} \phi d\nu = \int_E \left( \int_{C_\lambda} \phi dm_\lambda \right) d\mu(\lambda)$$

**Theorem 3.1.**  *$(E, w, \mu)$  satisfies the weighted B-M inequality if and only if  $(\bar{Z}, \nu)$  satisfies the B-M inequality.*

**Proof:** First (using the notation of lemma 2.1) we prove

**Lemma 3.1.**

$$\|P_d\|_{L^2(\nu)} = \|w^d G_d\|_{L^2(\mu)}$$

**Proof:**  $P_d$  is a homogeneous polynomial of degree  $d$  on  $\mathbb{C}^{N+1}$ . Now

$$\begin{aligned} \int_{\mathbb{C}^{N+1}} |P_d(t, z)|^2 d\nu &= \int_E \left( \int_{C_\lambda} |P_d(t, z)|^2 dm_\lambda \right) d\mu(\lambda) \\ &= \int_E |w^d G_d|^2 d\mu(\lambda), \text{ using (2.9)} \\ &= \|w^d G_d\|_{L^2(\mu)}^2 \quad \square \end{aligned}$$

Now, suppose  $(\bar{Z}, \nu)$  satisfies the B-M inequality. Applying that inequality to homogeneous polynomials, using lemmas 2.1 and 3.1 we obtain the weighted B-M inequality for  $(E, w, \mu)$ .

For the converse, suppose  $(E, w, \mu)$  satisfies the weighted B-M inequality.

We first note that if two monomials are of different degrees, they are orthogonal in  $L^2(\nu)$  since their restrictions to any  $C_\lambda$  are orthogonal in  $L^2(dm_\lambda)$ . Hence for a polynomial  $p$  on  $\mathbb{C}^{N+1}$ , written as a sum of homogeneous polynomials

$$(3.5) \quad p = \sum_{i=0}^d p_i \quad \text{then}$$

$$(3.6) \quad \|p\|_{L^2(\nu)} = \sum_{i=0}^d \|p_i\|_{L^2(\nu)}$$

Hence, for any  $\epsilon > 0$  there is a  $C > 0$  such that

$$(3.7) \quad \|p\|_Z \leq \sum_{i=0}^d \|p_i\|_Z \leq C(1 + \epsilon)^d \|p_i\|_{L^2(\nu)} \leq C(d + 1)(1 + \epsilon)^d \|p\|_{L^2(\nu)}$$

where the second inequality comes from lemmas 2.1, 3.1 and the weighted B-M inequality for  $(E, w, \mu)$ . The third inequality in (3.7) comes from (3.6). The B-M inequality for  $(\bar{Z}, \nu)$  follows from (3.7).  $\square$

**Coroallary 3.1.** *Suppose  $\bar{Z}$  is regular. Then  $(E, w, d\mu_{eq}(E, w))$  satisfies the weighted B-M inequality.*

**Proof:** It is a result of Nguyen-Zeriahi [NZ] combined with ([K], corollary 5.6.7) that  $(\bar{Z}, d\mu_{eq}(\bar{Z}))$  satisfies the B-M inequality. However, by theorem 2.2,  $\frac{1}{2\pi}d\mu_{eq}(\bar{Z}) = dm_\lambda \otimes d\mu_{eq}(E, w)$ .  $\square$

We will give another general situation in which the weighted B-M inequality holds (see also [StTo], theorem 3.2.3. (vi)).

**Theorem 3.2.** *Let  $E$  be a locally regular, compact set  $\subset \mathbb{R}^N$  and let  $w$  be continuous on  $E$  with  $\inf_{z \in E} w(z) > 0$ . Suppose that  $\sigma$  is a finite positive Borel measure on  $E$  and  $(E, \sigma)$  satisfies the B-M inequality. Then  $(E, w, \sigma)$  satisfies the weighted B-M inequality.*

**Proof:**  $\log w$  is continuous on  $E$  and so may be, by the Weierstrass theorem, approximated by (real) polynomials. That is, given  $\epsilon > 0$  there exists  $g_\epsilon = g_\epsilon(x_1, \cdot, x_n)$ , a real polynomial, such that  $\|\log w - g_\epsilon\|_E \leq \epsilon$ . Taking exponentials, we have

$$(3.9) \quad e^{-\epsilon} \leq \frac{w}{\exp(g_\epsilon)} \leq e^\epsilon \quad \text{for } z \in E$$

We consider  $g_\epsilon$  as a holomorphic polynomial

$$g_\epsilon = g_\epsilon(z_1, \dots, z_N)$$

Taking sufficient many terms in the power series for  $\exp(g_\epsilon)$  we get a holomorphic polynomial  $H$  such that, for  $\epsilon$  sufficiently small

$$(3.10) \quad 1 - 2\epsilon \leq \frac{w}{H} \leq 1 + 2\epsilon \quad \text{for } z \in E$$

Now, consider a weighted polynomial  $w^d G$  and let

$$(3.11) \quad J := GH^d$$

Then

$$(3.12) \quad w^d |G| = |J| \left( \frac{w}{|H|} \right)^d \quad \text{so that}$$

$$(3.13) \quad |J|(1 - 2\epsilon)^d \leq w^d |G| \leq |J|(1 + 2\epsilon)^d \quad \text{for } z \in E$$

Now, by the B-M inequality for  $(E, \sigma)$  we have, given  $\epsilon_1 > 0$  a constant  $C_1 > 0$  such that

$$(3.14) \quad \|J\|_E \leq C_1(1 + \epsilon_1)^{(h+1)d} \|J\|_{L^2(\sigma)}$$

where  $h := \deg H$ . Hence

$$\begin{aligned} \|w^d G\|_E &\leq \|J\|_E (1 + 2\epsilon)^d \quad \text{by the right inequality in (3.13)} \\ &\leq C_1(1 + \epsilon_1)^{(h+1)d} (1 + 2\epsilon)^d \|J\|_{L^2(\sigma)} \quad \text{by (3.14)} \\ &\leq C_1(1 + \epsilon_1)^{(h+1)d} \frac{(1 + 2\epsilon)^d}{(1 - 2\epsilon)^d} \|w^d G\|_{L^2(\sigma)} \quad \text{by} \end{aligned}$$

the left inequality in (3.13).

Now,  $\epsilon > 0$  having been chosen, and  $h$  fixed, we choose  $\epsilon_1$  so that  $(1 + \epsilon_1)^{(h+1)} \leq 1 + \epsilon$  and so we obtain

$$(3.16) \quad \|w^d G\|_E \leq C_1 \frac{(1 + \epsilon)^d (1 + 2\epsilon)^d}{(1 - 2\epsilon)^d} \|w^d G\|_{L^2(\sigma)}$$

But  $\epsilon > 0$  is arbitrary so the weighted B-M inequality holds.

**Example 3.1** Let  $B_R = \{x \in \mathbb{R}^N \mid |x| \leq R\}$  be the (real) ball of radius  $R$  (center the origin). Then  $(B_R, dx)$  satisfies the BM inequality (see [B]) where  $dx$  denotes Lebesgue measure.

Let  $w(x)$  be any continuous positive function on  $B_R$ . Then by theorem 3.2  $(B_R, w, dx)$  satisfies the weighted B-M inequality.

#### 4. $L^2$ THEORY OF WEIGHTED POLYNOMIALS

Let  $E$  be a compact non-pluripolar subset of  $\mathbb{C}^N$ ,  $w$  an admissible weight on  $E$ , and  $\mu$  a finite positive Borel measure with  $\text{supp}(\mu) = E$ . For  $d$  a positive integer, the monomials are linearly independent in  $L^2(w^{2d}\mu)$  ([Bl1], prop. 3.5 adapts to

this situation). Ordering via a lexicographic ordering on their multi-index exponents and applying the Gram-Schmidt procedure we obtain orthonormal polynomials  $\{p_\alpha^d(z, \mu)\}_{\alpha \in \mathbb{N}^n}$ . They satisfy

$$(4.1) \quad \int_{\mathbb{C}^N} p_\alpha^d(z, \mu) p_\beta^d(z, \mu) w^{2d} d\mu = \delta_{\alpha\beta}$$

for  $\alpha, \beta$  multi-indices.

We can write

$$(4.2) \quad p_\alpha^d(z, \mu) = a_\alpha^d z^\alpha + \text{(monomials of lower lexicographic order)} \text{ where } a_\alpha^d > 0$$

We will only consider these polynomials where  $|\alpha| = d$ .

In the case that  $(E, w, \mu)$  satisfies the weighted B-M inequality we will show that the leading exponents  $\{a_\alpha^d\}$  have asymptotic limits in the following sense. First we let

$$(4.3) \quad \Sigma_0 := \{\theta \in \mathbb{R}^N \mid \theta = (\theta_1, \dots, \theta_N), \sum_{j=1}^n \theta_j = 1, \theta_j > 0\}$$

We consider sequences of multi-indices  $\{\alpha(j)\}$  with, for some  $\theta \in \Sigma_0$

$$(4.4) \quad \lim_j |\alpha(j)| = +\infty \text{ and } \lim_j \frac{\alpha(j)}{|\alpha(j)|} = \theta$$

**Theorem 4.1.** *Suppose  $(E, w, \mu)$  satisfies the weighted B-M inequality and  $\{\alpha(j)\}$  is a sequence of multi-indices satisfying (4.4). Then*

$$\lim_j \left( a_{\alpha(j)}^d \right)^{\frac{1}{d}} = \frac{1}{\tau^w(E, \theta)}$$

where  $\tau^w(E, \theta)$  is the weighted directional Tchebyshev constant of  $E$  in the direction  $\theta$ .

**Proof:** First, we recall the definition of weighted directional Tchebyshev constant (see [BL2]). For  $\alpha$  a multiindex we let  $P(\alpha) = \{q \mid q = z^\alpha + \sum_{\beta < \alpha} c_\beta z^\beta\}$  where

$c_\beta \in \mathbb{C}$  and the notation  $\beta < \alpha$  is used to denote the fact that the multiindex  $\beta$  precedes  $\alpha$  in the lexicographic ordering on the multi-indices.

For  $\alpha$  a multiindex with  $|\alpha| = d$  we let  $t_\alpha^d$  denote a (Tchebyshev) polynomial which minimizes  $\{\|w^d q\|_E \mid q \in \mathcal{P}(\alpha)\}$ . That is,  $t_\alpha^d \in \mathcal{P}(\alpha)$  and

$$(4.5) \quad \|w^d t_\alpha^d\|_E = \inf\{\|w^d q\|_E \mid q \in \mathcal{P}(\alpha)\}.$$

Then (see [BL2]) it is known that for a sequence of multi-indices  $\{\alpha(j)\}_{j=1,2,\dots}$  satisfying (4.4) the limit

$$(4.6) \quad \tau^w(E, \theta) := \lim_j \|w^d t_{\alpha(j)}^d\|_E^{\frac{1}{d}}$$

exist and is called the weighted Tchebyshev constant in the direction  $\theta \in \Sigma_0$ .

Now, it follows from general Hilbert space theory that

$$(4.7) \quad a_\alpha^d = \frac{1}{\|w^d q_\alpha^d\|_{L^2(\mu)}}$$

where  $q_\alpha^d$  is the unique polynomial in  $\mathcal{P}(\alpha)$  satisfying.

$$(4.8) \quad \|w^d q_\alpha^d\|_{L^2(\mu)} = \inf\{\|w^d q\|_{L^2(\mu)} \mid q \in \mathcal{P}(\alpha)\}$$

Now, for  $\epsilon > 0$ , there is a  $C > 0$  such that

$$(4.9) \quad \begin{aligned} \|w^d q_\alpha^d\|_E &\leq C(1 + \epsilon)^d \|w^d q_\alpha^d\|_{L^2(\mu)} \quad \text{by the weighted B-M inequality} \\ &\leq C(1 + \epsilon)^d \|w^d t_\alpha^d\|_{L^2(\mu)} \quad \text{by (4.8)} \\ &\leq C_1(1 + \epsilon)^d \|w^d t_\alpha^d\|_E \end{aligned}$$

since the sup norm estimates the  $L^2$  norm for a finite measure with compact support.

Hence, for every  $\epsilon > 0$  there is a constant  $C_1 > 0$  such that

$$(4.10) \quad \|w^d t_\alpha^d\|_E \leq \|w^d q_\alpha^d\|_E \leq C_1(1 + \epsilon)^d \|w^d t_\alpha^d\|_E$$

Now, given a sequence of multi-indices  $\{\alpha(j)\}$  satisfying (4.4), taking the  $1/d$  powers of the expressions in (4.10), letting  $j \rightarrow \infty$ , using (4.6), (4.7) and the fact that  $\epsilon > 0$  is arbitrary, the result follows.  $\square$

**Example 4.1** On  $\mathbb{R}^N$  we consider orthonormal polynomials with respect to the inner product given by  $e^{-H(x)}dx$  where  $dx$  is Lebesgue measure on  $\mathbb{R}^N$  and  $H(x)$  satisfies

- i)  $H(x)$  is homogeneous of degree  $\gamma > 0$ . That is
- $$(4.11) \quad H(cx) = c^\gamma H(x) \quad c \in \mathbb{R}.$$
- ii)  $H(x) > 0$  for all  $x \neq 0$

We let  $\{p_\alpha(x)\}_{\alpha \in \mathbb{N}^N}$  denote the orthonormal polynomials obtained, by applying the Gram-Schmidt procedure to the (real) monomials ordered via a lexicographic ordering of their exponents. Then

$$(4.12) \quad \int_{\mathbb{R}^N} p_\alpha(x) p_\beta(x) e^{-H(x)} dx = \delta_{\alpha\beta}$$

for any two multi-indices  $\alpha, \beta$ .

We write

$$(4.13) \quad p_\alpha(x) = a_\alpha x^\alpha + (\text{sum of monomials of lower lexicographic order}). \quad a_\alpha > 0$$

We will obtain asymptotic estimates (see (4.24)) for  $|a_\alpha|^{\frac{1}{|\alpha|}}$  for a sequence of multi-indices satisfying (4.4). In the case  $N = 1$ , these estimates are Theorem VII, 1.2 of [SaTo]. In that case explicit knowledge of the set  $S_w$  (defined below) yields an explicit form to the right hand side of (4.24). It would be of interest to find  $S_w$  explicitly in the case  $N > 1$ .

In the one-dimensional case ( $N = 1$ ) this gives a version of so-called weak asymptotics and in this case considerably more detailed asymptotic results are known (see [SaTo] or [Dei]).

From general Hilbert space theory,

$$(4.14) \quad a_\alpha^{-1} = \inf \{ \|e^{-\frac{H(x)}{2}} q(x)\|_{L^2(\mathbb{R}^N)} \mid q \in \mathcal{P}_\mathbb{R}(\alpha) \}$$

where  $q \in \mathcal{P}_\mathbb{R}(\alpha) = \{\text{polynomials of the form } x^\alpha + \sum_{\beta < \alpha} r_\beta x^\beta \text{ with } r_\beta \in \mathbb{R}\}$ .

For  $|\alpha| = d$  we scale by  $x = d^{\frac{1}{\gamma}} y$ . We get

$$(4.15) \quad a_\alpha^{-1} = d^{\frac{d}{\gamma}} \inf \left\{ \left\| e^{-\frac{dH(y)}{2}} q(y) \right\|_{L^2(\mathbb{R}^N)} \mid q \in \mathcal{P}_\mathbb{R}(\alpha) \right\}.$$

Consider the weight  $w(y) = e^{-\frac{H(y)}{2}}$  on  $\mathbb{R}^N \subset \mathbb{C}^N$ . This weight is admissible in the sense of ([SaTo], appendix B) although, since  $\mathbb{R}^N$  is not compact, not in the sense of 1.10. We let  $Q(y) = \frac{H(y)}{2}$ . The following is known ([SaTo], appendix B).  $S_w := (dd^c V_{\mathbb{R}^N, Q})^N$  has compact support. For any weighted polynomial  $w^d p$  we have

$$(4.16) \quad |w^d p(y)| \leq \|w^d p\|_{S_w} \exp(d(V_{\mathbb{R}^N, Q} - Q))$$

In particular

$$(4.17) \quad \sup_{\mathbb{R}^N} |w^d p| = \sup_{S_w} |w^d p|.$$

Now  $V_{\mathbb{R}^N, Q} \in \mathcal{L}$  so, fixing  $R > 0$  large, using (4.16), there is a constant  $A > 0$  such that.

$$(4.18) \quad |w^d p(y)| \leq \|w^d p\|_{S_w} e^{-Ad|y|^\gamma} \text{ for } |y| \geq R$$

Now

$$(4.19) \quad \|w^d p\|_{L^2(\mathbb{R}^N)}^2 \leq \|w^d p\|_{L^2(B_R)}^2 + \|w^d p\|_{S_w}^2 \int_{|y| \geq R} e^{-2Ad|y|^\gamma} dy$$

We may assume  $S_w \subset B_R$ . Then

$$(4.20) \quad \begin{aligned} \|w^d p\|_{S_w} &= \|w^d p\|_{B_R} \leq C(1 + \epsilon)^d \|w^d p\|_{L^2(B_R)} \text{ since} \\ (B_R, w, dx) &\text{ satisfies the weighted B-M inequality (see example 3.1)} \end{aligned}$$

Now simple estimates show there is a constant  $c_1 > 0$  such that

$$(4.21) \quad \int_{|y| \geq R} e^{-2Ad|y|^\gamma} dy \leq e^{-dc_1}$$

We get

$$(4.22) \quad \|w^d p\|_{L^2(\mathbb{R}^N)} \leq \|w^d p\|_{L^2(B_R)} \left(1 + \frac{C^2(1+\epsilon)^{2d}}{e^{dc_1}}\right)^{\frac{1}{2}}.$$

However for  $\epsilon > 0$  sufficiently small, the expression on the right of (4.22) is bounded in  $d$ . It follows that for a sequence of multi-indices  $\{\alpha(j)\}$  satisfying (4.4)

$$(4.23) \quad \lim_{j \rightarrow \infty} [\inf\{\|w^d q\|_{L^2(\mathbb{R}^N)} \mid q \in \mathcal{P}_{\mathbb{R}}(\alpha)\}^{\frac{1}{d}}] = \lim_{j \rightarrow \infty} [\inf\{\|w^d q\|_{L^2(B_R)} \mid q \in \mathcal{P}_{\mathbb{R}}(\alpha(j))\}^{\frac{1}{d}}]$$

But, by (the proof of) theorem 4.1 the limit on the right side of (4.23) exists and it may be identified with  $\tau^w(S_w, \theta)$  using (4.1). Hence we obtain

$$(4.24) \quad \lim_{j \rightarrow \infty} a_{\alpha(j)}^{\frac{1}{d}} d^{\frac{1}{\gamma}} = \frac{1}{\tau^w(S_w, \theta)}$$

## 5. GENERAL CIRCULAR SETS

The circular sets which arise in the form  $\hat{\hat{Z}}(E, w)$  (i.e. the polynomially convex hull of a set of the form  $Z(E, w)$ ) are

- i) polynomially convex
- ii) circular
- iii) compact
- iv) non-pluripolar

However, they are not the most general sets with the properties i), ii), iii), iv). We will show, however, that the most general set with those properties is, in an appropriate sense, a limit of sets of the form  $\hat{\hat{Z}}(E, w)$

Let  $Z \subset \mathbb{C}^{N+1}$  be a set with properties i), ii), iii), iv) above. Then the origin is an interior point of  $Z$ . We associate to  $Z$  the a function on  $\mathbb{C}^N$  defined by

$$(5.1) \quad w(\lambda) := \sup\{|t| \mid (t, z) \in Z \cap C_\lambda\}$$

Then  $w(\lambda) > 0$  for all  $\lambda$  and  $w$  is bounded above. We let  $Q(\lambda) := -\log w(\lambda)$ .

**Proposition 5.1.**  *$w$  is u.s.c. on  $\mathbb{C}^N$ .*

**Proof:** Fix  $\lambda^0 \in \mathbb{C}^N$ . Let  $\{\lambda^s\}_{s=1,2,\dots}$  be a sequence in  $\mathbb{C}^N$  converging to  $\lambda^0$ . We may suppose, passing to a subsequence if necessary, that  $\lim_s(\lambda^s) := w^0$  exists. The points  $(w(\lambda^s), \lambda_1^s w(\lambda^s), \dots, \lambda_N^s w(\lambda^s)) \in Z \cap C_{\lambda^s}$  so, since  $Z$  is compact, the point  $(w^0, \lambda_1^0 w^0, \dots, \lambda_N^0 w^0) \in Z \cap C_{\lambda^0}$ . Thus, by definition of  $w$ ,  $w(\lambda^0) \geq w^0 = \lim_s w(\lambda^s)$ .  $\square$

**Proposition 5.2.**  *$Q^*$  is PSH on  $\mathbb{C}^N$ .*

**Proof:**  $Z = \{(t, z) \in \mathbb{C}^{N+1} \mid H_Z(t, z) \leq 0\}$

Now

$$H_Z(t, z) = \log |t| + H_Z(1, \lambda) \text{ so that}$$

$$\log w(\lambda) = -H_Z(1, \lambda) \text{ and}$$

$$Q(\lambda) = H_Z(1, \lambda)$$

But  $H_Z = H_Z^*$  outside a circular pluripolar set in  $\mathbb{C}^{N+1}$  so by [BL2, lemma 6.1]  $H_Z(1, \lambda) = H_Z^*(1, \lambda)$  q.e. on  $\mathbb{C}^N$ .

Thus  $Q^* = H_Z^*(1, \lambda) \quad \square$ .

**Example 5.1**  $Z = \{|t|^2 + |z_1|^2 + \dots + |z_N|^2 \leq 1\} \subset \mathbb{C}^{N+1}$ . Then  $w(\lambda) = (1 + |\lambda_1|^2 + \dots + |\lambda_N|^2)^{-\frac{1}{2}}$  and  $Q(\lambda) = \frac{1}{2} \log(1 + |\lambda_1|^2 + \dots + |\lambda_N|^2)$ .

As the above example illustrates, in general the functions  $w(\lambda)$  which arise in this way are *not* admissible weights in the sense of [SaTo], appendix B, definition 2.1. In particular,  $\lim_{|\lambda| \rightarrow \infty} |\lambda| w(\lambda) \neq 0$

Let  $Z_R := \{(t, z) \in Z \mid \frac{|z|}{|t|} \leq R\}$ .

**Proposition 5.3.**  $\lim_{R \rightarrow \infty} d\mu_{eq}(Z_R) = d\mu_{eq}(Z)$  weak\*

**Proof:**  $Z_R \cup T \uparrow Z \cup T$  where  $T$  is the pluripolar set  $\{(t, z) \in \mathbb{C}^{N+1} \mid t = 0\}$ . Hence  $V_{Z_R}^* \downarrow V_Z^*$  as  $R \rightarrow \infty$ . The result follows from ([K], cor 5.2.5 and 5.2.6) and the continuity of the Monge-Ampère operator under decreasing limits [K].  $\square$

**Proposition 5.4.**  $\lim_{R \rightarrow \infty} L_*\left(\frac{1}{2\pi}d\mu_{eq}(Z_R)\right) = (dd^c Q^*)^N$

**Proof:** Let  $w_R = w|_{B(0,R)}$ . Let  $Q_R = -\log w_R$ . Then the family of PSH functions  $V_{B(0,R),Q_R}^*$  is decreasing as  $R \rightarrow \infty$ . Since  $Q^*$  is PSH,  $V_{B(0,R),Q_R}^* = Q^*$  on  $B(0,R)$  so  $V_{B(0,R),Q_R}^* \downarrow Q^*$ . The result follows using theorem 2.2.  $\square$ .

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